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Potential symmetries of the nonlinear wave equation $u_{tt} = (uu_x)_x$ and related exact and approximate solutions

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Abstract

This paper explores potential symmetries of the nonlinear wave equation $u_{tt} = (uu_x)_x$, as well as related new similarity reductions and exact solutions of this equation. New approximate solutions of the perturbed nonlinear equations stemming from the exact solutions of the equation are obtained by applying a new approach to the use of the Lie group technique for differential equations dependent on a small parameter. In addition, some nonlinear wave equations exactly reducible to the equation $u_{tt} = (uu_x)_x$ are constructed using this approach.

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1. Introduction

Lie group theory provides a powerful tool for obtaining analytical solutions of a large class of partial differential equations (PDEs). The most effective and universal method designed for this purpose is the symmetry reduction procedure (see, e.g., [1, 2]). The invariance of a PDE (or a system of PDEs) under a Lie group of point transformations is used to construct special solutions, which are invariant under some subgroup of the full group admitted by the equation (similarity or invariant solutions). The conditional symmetry approach (nonclassical method, [3,4]) may be applied to enlarge the class of solutions obtainable by the symmetry reduction method. Additionally, new solutions, not obtainable through the classical and nonclassical Lie algorithms, may be arrived at by developing certain generalizations of the nonclassical method (e.g., [5–8]). The classical Lie group method based on the invariance of PDEs under point transformations (point symmetries) can be further extended by considering the invariance under contact transformations (contact symmetries) and Lie–Bäcklund transformations (Lie–Bäcklund symmetries) (see, e.g., [1]). Combinations of these extended symmetries with the conditional symmetry approach are also possible (e.g., [9, 10]).

Recently, Bluman *et al* [1,11,12] introduced a method for finding a new class of symmetries for a system of PDEs for the case where at least one of the PDEs can be written in a conserved form. These new *potential* symmetries, determined as point (or Lie–Bäcklund) symmetries of the associated auxiliary system arising from conserved forms by the introduction of additional (potential) variables, are nonlocal symmetries of the original PDEs whose infinitesimals depend on the integrals of the original dependent variables (potentials). A potential symmetry enables the construction of solutions of the original system of PDEs which cannot be obtained as invariant solutions of its local symmetries.

The purpose of this paper was threefold.

First we wished to explore the potential symmetries and related new exact solutions of the equation

$$u_{tt} = (uu_x)_x \tag{1.1}$$

which arises in several different physical contexts (e.g., longitudinal wave propagation on a moving threadline, electromagnetic transmission line, transonic equation [13], dynamics of a finite nonlinear string [14]).

The second objective of this paper was to develop a new approach to the use of the Lie group technique for differential equations depending on a small parameter. Our aim was to be able to construct equations that could be reduced by exact transformations to an unperturbed equation and at the same time would coincide approximately (within some range of the equation parameter) with the initial (perturbed) equation. The new method differs conceptually from the symmetry group methods outlined above. The central concept underpinning the latter is the symmetry of the equation, which is defined as a group of transformations that leaves the equation invariant and consequently maps any solution to another solution of the equation. Our approach does away with the invariance requirement while using the Lie group machinery. For a perturbed differential equation depending on a small parameter ϵ, the invariance requirement is replaced by the requirement that the unperturbed equation transform infinitesimally (for small values of the group parameter a) into the perturbed equation with $\epsilon = a$. The infinitesimal Lie technique modified with this requirement yields determining equations for the group generators that differ from those of the symmetry group method. The corresponding infinitesimal transformations map any solution of the unperturbed equation to an approximate (valid up to first order in ϵ) solution of the perturbed equation. The finite transformations defined on the basis of the group generators, as a solution of the corresponding Cauchy problem, are used to arrive at a new equation depending on the group parameter a. This equation, which for $a = \epsilon \ll 1$ coincides with the original perturbed equation, can be converted into the unperturbed equation by the exact transformation. Thus, the method developed allows: (i) extending any solution of the unperturbed equation to the approximate solution of the perturbed equation, and (ii) finding new integrable equations that have (at least, in some parameter interval) a definite physical meaning.

Several symmetry based perturbation methods have been developed recently. The approach developed in a series of papers by Baikov *et al* (see, e.g., [15, 16]), referred to as the approximate symmetry group method, represents a perturbation technique embedded into the standard procedure of the classical Lie group method. The approach developed by Fushchich and Shtelen [17] combines a common perturbation technique with the symmetry group method: at the first stage, a perturbation technique is applied to approximately replace the original equations by the system of equations for the zero- and first-order parts, after which, at the second stage, the usual symmetry group method is applied to obtain solutions of this coupled system. Both methods [15] and [17] are based on the symmetry of the equations, so that the invariance requirement (the approximate invariance of the original equation in [15] and the exact invariance of the system approximating the original equation in [17]) is a central feature of the methods. Thus, our method, in which the invariance is replaced by another requirement, differs conceptually from [15] and [17].

Discovering related differential equations, one with a definite physical meaning and the other of simpler form—which was another goal of our method—also figures among applications of symmetry methods to differential equations. It is usually implemented by comparing the symmetry groups of a given differential equation and another differential equation (target equation) [1]; here the symmetry of the equations again plays a central role. Our method, on the other hand, operates on the Lie group of transformations that do not leave equations invariant but transform one equation into another.

The third objective of this paper was to apply the approach described above to perturbed nonlinear wave equations having equation (1.1) as an unperturbed part. We have found new approximate solutions of the perturbed nonlinear wave equations, in particular, solutions stemming from the exact solutions of equation (1.1) defined in this paper via potential symmetries. As another result of applying the new method we have constructed some nonlinear wave equations that are reducible to equation (1.1) by exact transformations and at the same time coincide approximately with the initial perturbed equations.

The paper is organized as follows.

In section 2, we study potential symmetries of equation (1.1) and define related similarity solutions. In section 3, we develop the new approach to use the Lie group technique for differential equations depending on a small parameter. In section 4, the new approach is applied to some nonlinear perturbed equations having equation (1.1) as an unperturbed part. New approximate solutions of these perturbed equations are obtained and some nonlinear wave equations exactly reducible to equation (1.1) are constructed as a result of application of the approach. Section 5 contains comments on the approach developed and prospects for future work.

2. Potential symmetries of equation (1.1) and related exact solutions

In [1] Bluman and Kumei describe the method to enlarge the classes of symmetries of differential equations. By writing a given PDE $R\{x, t, u\}$ in a conserved form an auxiliary system $S\{x, t, u, v\}$ with potentials v as auxiliary variables is constructed. Any Lie group of point transformations admitted by $S\{x, t, u, v\}$ induces a symmetry admitted by $R\{x, t, u\}$, and if at least one of the infinitesimals of variables (x, t, u) of $S\{x, t, u, v\}$ depends explicitly on v, then the corresponding symmetry of $R\{x, t, u\}$ is a potential (nonlocal) symmetry which is not a point symmetry of $R\{x, t, u\}$. This, in particular, enables one to construct solutions of a given PDE $R\{x, t, u\}$, which cannot be obtained as invariant solutions of its local symmetries, since, in general, invariant solutions of $S\{x, t, u, v\}$ arising from its point symmetries yield solutions of $R\{x, t, u\}$ that are not invariant solutions of any point symmetry admitted by $R\{x, t, u\}$. Another application of potential symmetries is their use in the linearization of nonlinear PDEs by a non-invertible mapping.

In order to find the potential symmetries of equation (1.1) we write it in a conserved form

$$D_x(F) - D_t(G) = 0 \tag{2.1}$$

where

$$F = uu_x \qquad G = u_t. \tag{2.2}$$

The associated auxiliary system $S\{x, t, u, v\}$ is given by

$$v_x = u_t \qquad v_t = uu_x. \tag{2.3}$$

A Lie point symmetry admitted by $S\{x, t, u, v\}$ is a symmetry characterized by an infinitesimal generator of the form

$$X = \xi(x, t, u, v)\frac{\partial}{\partial x} + \tau(x, t, u, v)\frac{\partial}{\partial t} + \phi_1(x, t, u, v)\frac{\partial}{\partial u} + \phi_2(x, t, u, v)\frac{\partial}{\partial v}. \tag{2.4}$$

The associated Lie algebra is infinite-dimensional and it is spanned by

$$\begin{aligned} &X_1 = (xv + tu^2)\frac{\partial}{\partial x} + (xu + 2tv)\frac{\partial}{\partial t} - 2uv\frac{\partial}{\partial u} - \left(\frac{2}{3}u^3 + \frac{3}{2}v^2\right)\frac{\partial}{\partial v} \\ &X_2 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} \qquad X_3 = \frac{\partial}{\partial x} \qquad X_4 = -\frac{t}{2}\frac{\partial}{\partial t} + u\frac{\partial}{\partial u} + \frac{3}{2}v\frac{\partial}{\partial v} \\ &X_5 = \frac{\partial}{\partial t} \qquad X_6 = \frac{\partial}{\partial v} \qquad X_\infty = f(u, v)\frac{\partial}{\partial u} + g(u, v)\frac{\partial}{\partial v} \end{aligned} \tag{2.5}$$

where the functions $f(u, v)$ and $g(u, v)$ satisfy the equations

$$f_u - ug_v = 0 \qquad f_v - g_u = 0. \tag{2.6}$$

The infinitesimal operators X_2, X_3, X_4, X_5, X_6 are projectable to the space (x, t, u) and X_2, X_3, X_4, X_5 projects onto point symmetries of (1.1). The infinitesimal operators X_1 and X_∞ are not projectable; they define the desired potential symmetries.

The operator X_∞ generates an infinite-parameter group (subgroup) of transformations and thus leads to the invertible mapping for the system (2.3) which in turn leads to the non-invertible mapping of (1.1). Employing the procedure for determining such invertible mappings given in [1] (theorems 6.4.1-1 and 6.4.1-2) leads to the hodograph transformation:

$$z_1 = u \qquad z_2 = v \qquad W_1 = x \qquad W_2 = t \tag{2.7a}$$

$$\frac{\partial W_1}{\partial z_2} - \frac{\partial W_2}{\partial z_1} = 0 \qquad \frac{\partial W_1}{\partial z_1} - z_1\frac{\partial W_2}{\partial z_2} = 0. \tag{2.7b}$$

As in the case of point symmetries, potential symmetries may be used to derive similarity reductions (solutions) of the initial equation $R\{x, t, u\}$. We will investigate similarity reductions associated with the point symmetry X_1 of (2.3) which is a potential symmetry of (1.1). The corresponding characteristic equations can be written as

$$\frac{\mathrm{d}x}{xv + tu^2} = \frac{\mathrm{d}t}{xu + 2tv} = -\frac{\mathrm{d}u}{2uv} = -\frac{\mathrm{d}v}{\frac{2}{3}u^3 + \frac{3}{2}v^2}. \tag{2.8}$$

If we introduce the new variables

$$q = u^{3/2} \qquad \lambda = u^{1/2}x - ut \qquad \mu = u^{1/2}x + ut \tag{2.9}$$

then solving the transformed characteristic equations yields the following three similarity variables

$$\vartheta = \lambda\mu = ux^2 - u^2t^2 \tag{2.10}$$

and

$$w_1 = \frac{9}{16}\frac{v^2}{q} - \frac{q}{4} \qquad w_2 = \frac{\mu^2}{4}(2q + 3v + 4w_1). \tag{2.11}$$

Solving equations (2.11) for q and v and treating ϑ as an independent variable and w_1 and w_2 as functions of ϑ, we obtain the similarity reduction in the form

$$q = \frac{1}{w_2(\vartheta)}\left(\frac{w_2(\vartheta)}{\mu} - w_1(\vartheta)\mu\right)^2 \qquad v = \frac{2}{3w_2(\vartheta)}\left(\frac{w_2(\vartheta)^2}{\mu^2} - w_1(\vartheta)^2\mu^2\right) \tag{2.12}$$

where μ and ϑ are defined by (2.9) and (2.10).

To obtain the corresponding system of ODEs for the functions $w_1(\vartheta)$ and $w_2(\vartheta)$ one has to substitute this similarity form into the original system of equations (2.3). That, however, will entail a very complicated algebra since both the similarity variable (2.10) and the right-hand sides of the forms (2.12) include the dependent variable u. Simplifications are achieved if we transform the original equations (2.3) to the variables q, v, λ and μ. After lengthy but straightforward calculations we arrive at the system of equations for $q(\lambda, \mu)$ and $v(\lambda, \mu)$ replacing the original system (2.3) in the form

$$q[(3v + 2q)_\lambda + (3v - 2q)_\mu] + 2(\lambda - \mu)(v_\lambda q_\mu - v_\mu q_\lambda) = 0 \tag{2.13a}$$

$$q[(3v - 2q)_\mu - (3v + 2q)_\lambda] + (\lambda + \mu)(v_\lambda q_\mu - v_\mu q_\lambda) = 0. \tag{2.13b}$$

Substituting the reduction (2.12) into equations (2.13) leads to the system of equations for $w_1(\vartheta)$ and $w_2(\vartheta)$ in the form

$$w_2' - \tfrac{1}{3}\vartheta w_1' = 0 \tag{2.14a}$$

$$\vartheta\left(\frac{w_1^2}{w_2}\right)' + 2\left(\frac{w_1^2}{w_2}\right) - \frac{1}{3}w_1' = 0 \tag{2.14b}$$

where primes indicate derivatives with respect to ϑ. This system is readily solved to give

$$w_1 = I\vartheta^{-1}(w_2^2 + b_1 w_2)^{1/2} \qquad I = \pm 1 \tag{2.15a}$$

and

$$\vartheta = b_2(w_2^2 + b_1 w_2)^{1/2}\left[w_2 + \frac{b_1}{2} + (w_2^2 + b_1 w_2)^{1/2}\right]^{-3I} \tag{2.15b}$$

where b_1 and b_2 are constants and $I = 1$ and -1 correspond to two different solutions.

In addition, there exists a solution not covered by (2.15), namely

$$w_1 = 0 \qquad w_2 = b_2 = \text{const.} \tag{2.16}$$

However, the corresponding solution of (1.1) given by $u = b_0(x + \sqrt{u}t)^{-4/5}$ is not of interest, since it is a particular case of solutions obtained by considering characteristic curves of equation (1.1).

The relations (2.15a) and (2.15b) together with (2.12), (2.9) and (2.10) define the two ($I = \pm 1$) solutions of equation (1.1) in a parametric form with w_2 being a parameter. The parameter w_2 is eliminated with the use of (2.15a) and the first relation of (2.12) to obtain

$$w_2 = \frac{\mu^2 q}{(\mu^2 - \lambda^2)^2}\left[\lambda^2 + I\left(\lambda^2\mu^2 - b_1\frac{\mu^2 - \lambda^2}{q}\right)^{1/2}\right]^2. \tag{2.17}$$

Substituting (2.9) and (2.10) into (2.17) and (2.15b) after some algebra yields

$$ux^2 - u^2t^2 = b_2\alpha\beta\left(\alpha^2 + \frac{b_1}{2} + \alpha\beta\right)^{-3I} \tag{2.18a}$$

where

$$\begin{Bmatrix}\alpha\\ \beta\end{Bmatrix} = \left|\frac{1}{4u^{1/4}xt}(x \pm u^{1/2}t)[(u^{1/2}x \mp ut)^2 + R]\right| \qquad R = I[(ux^2 - u^2t^2)^2 - 4b_1xt]^{1/2}. \tag{2.18b}$$

The relations (2.18) define two families ($I = \pm 1$) of the implicit form solutions of equation (1.1) depending on two arbitrary constants b_1 and b_2. It is seen from (2.18b) that b_1 should be negative. Note that the solutions can be enriched by using the point symmetries of (1.1); in particular, t and x can be replaced by $t + t_0$ and $x + x_0$ (where t_0 and x_0 are arbitrary constants) to avoid singularities in the initial and boundary conditions.

The particular cases of solutions (2.18) corresponding to $b_1 = 0$ can be represented as

$$x^2 = ut^2 + K\left(\frac{t}{u}\right)^{4/5} \qquad (I = 1) \tag{2.19}$$

and

$$t^2 = \frac{x^2}{u} - C\left(\frac{x}{u^2}\right)^{8/7} \qquad (I = -1) \tag{2.20}$$

where K and C are constants ($K = (2b_2)^{1/5}, C = (b_2/32)^{-1/7}$). The formulae (2.19) and (2.20) may be also obtained straight from equations (2.15) and (2.12). It is worth noting here that despite the fact that the derivation of the solutions for the general case $b_1 \neq 0$ relies on the use of variables including $\sqrt{u}$ and therefore its validness is restricted by the domain of hyperbolicity of (1.1) $u > 0$, the solutions (2.19) and (2.20) not containing $\sqrt{u}$ are also valid in the domain $u < 0$.

It is pointed out in [18] that a wider class of solutions may be obtained by direct introduction of the similarity form for u into the original equation. However, due to the structure of the similarity form, defined by the first relation of (2.12) together with (2.9) and (2.10), that includes the dependent variable u through both μ and the similarity variable ϑ, substitution of (2.12) into (1.1) involves a tedious algebra which is virtually unmanageable even with symbolic manipulation programs. One might expect that a progress could be achieved by using the variables q, λ and μ instead of u, x and t. However, the overdetermined system of four ODEs for two functions $w_1(\vartheta)$ and $w_2(\vartheta)$, obtained by substituting the first relation of (2.12) into equation (1.1) transformed to the variables q, λ and μ is still too complicated to find closed-form solutions.

The solutions (2.18)–(2.20) found in this section with the use of the potential symmetry associated with the generator X_1 represent new exact closed-form solutions of the initial equation (1.1) These solutions are not obtainable by the use of the point symmetries of equation (1.1). They do not figure among solutions invariant under conditional symmetries considered in [19]. They, being obtained via the potential symmetry X_1, differ from the solutions found in [18] with the use of the potential symmetry $X = X_2 + X_\infty$ with $f = v$ and $g = u$. They cannot be obtained by the direct method of Clarkson and Kruskal [20] or by its recent extensions [21–23], since all these methods start from similarity forms, in which the independent similarity variable ϑ does not depend on u.

In conclusion of this section, we will make some comments on the possible usefulness of the solutions found. As it was mentioned above, the wave equation (1.1) arises in a number of different physical contexts. It is also of interest in that it represents one of the simplest examples of a class of nonlinear hyperbolic PDEs displaying the 'blow-up' phenomenon, which usually manifests itself by derivatives of u of a certain order becoming infinite at some point x. Equations (2.18)–(2.20), which include arbitrary parameters, in point of fact, describe a variety of solutions. Depending on a choice of parameters (which also specifies the initial data), they can produce both globally smooth solutions and solutions with the first derivative u_x becoming infinite at some point, while u itself stays finite. The solutions are also applicable after the time when the singularity forms; they describe propagating discontinuities at which u_x is infinite but u is finite. These exact solutions can help provide an understanding of the complicated phenomena which is difficult to achieve with the use of a numerical method. They differ in this respect from many other analytical solutions of the nonlinear wave equations obtained by the symmetry group methods—mainly due to the fact that they include u into both the dependent and the independent similarity variables, which provides more possibilities for describing complicated features.

The exact solutions of equation (1.1) found in this section are used in section 4 for constructing new approximate solutions of the perturbed nonlinear wave equations.

3. A new approach to the use of the Lie group technique for differential equations with a small parameter

We will consider a kth-order scalar differential equation depending on a small parameter ϵ, namely

$$\Delta(z, u_{(1)}, u_{(2)}, \ldots, u_{(k)}; \epsilon) = \Delta_0(z, u_{(1)}, u_{(2)}, \ldots, u_{(k)}) + \epsilon\Delta_1(z, u_{(1)}, u_{(2)}, \ldots, u_{(k)}) = 0. \tag{3.1}$$

We use the following notation: $z = (x, u) = (x^1, x^2, \ldots, x^n, u)$ denotes the vector, in which $x = (x^1, x^2, \ldots, x^n)$ are n independent variables and u is the dependent variable, and $u_{(j)}$ denotes the set of all jth-order partial derivatives of u with respect to x.

The approach developed allows, in particular, finding approximate solutions of such equations. Some symmetry based perturbation methods have been developed recently to find approximate solutions of nonlinear differential equations. To show the difference between those methods and our approach we will outline the approximate symmetry group method due to Baikov *et al* [15, 16] and the approximate symmetry approach of Fushchich and Shtelen [17] before presenting our approach.

3.1. The symmetry based perturbation methods

In the approximate symmetry group method of Baikov *et al* [15], the one-parameter (a) Lie group of transformations

$$z^* = F(z; \epsilon; a) \approx F_0(z; a) + \epsilon F_1(z; a) \tag{3.2}$$

depending on a small parameter ϵ is considered. Let

$$X = \zeta(z; \epsilon)\frac{\partial}{\partial z} \approx X_0 + \epsilon X_1 \qquad \zeta(z; \epsilon) \approx \zeta_0(z) + \epsilon\zeta_1(z) \tag{3.3a}$$

$$\zeta = (\xi, \eta) = (\xi^1, \xi^2, \ldots, \xi^n, \eta) \tag{3.3b}$$

be the infinitesimal generator of (3.2). Then the transformations (3.2) form an approximate symmetry group of equation (3.1) if

$$X^{(k)}\Delta(z, u_{(1)}, u_{(2)}, \ldots, u_{(k)}; \epsilon)|_{\Delta=\mathrm{O}(\epsilon^2)} = \mathrm{O}(\epsilon^2) \tag{3.4}$$

where

$$X^{(k)} \approx X_0^{(k)} + \epsilon X_1^{(k)} \tag{3.5}$$

is the kth extended infinitesimal generator of (3.3). If in (3.3) the generator $X_0 \neq 0$, it represents an exact symmetry of the unperturbed equation $\Delta_0 = 0$ but, in general, not any operator X_0 admitted by the unperturbed equation is inherited by the perturbed equation. The *approximate invariant solutions* are constructed from the approximate similarity variables obtained by solving (up to first order in ϵ) the approximate characteristic equations

$$\frac{\mathrm{d}x^1}{\xi_0^1(x, u) + \epsilon\xi_1^1(x, u)} = \cdots = \frac{\mathrm{d}x^n}{\xi_0^n(x, u) + \epsilon\xi_1^n(x, u)} = \frac{\mathrm{d}u}{\eta_0(x, u) + \epsilon\eta_1(x, u)}. \tag{3.6}$$

Thus, the approximate symmetry group method represents a perturbation technique embedded into the standard procedure of the classical Lie group symmetry method.

The approach developed by Fushchich and Shtelen in [17] to obtain approximate solutions of the perturbed nonlinear wave equation represents another combination of a common perturbation technique with the symmetry group method. It starts with applying a perturbation technique to reduce the original equation (3.1) to the coupled system of equations for the zero-order and first-order parts of the solution, as follows:

$$u = w + \epsilon v \tag{3.7}$$

$$\begin{aligned} &\Delta_0(z, w_{(1)}, w_{(2)}, \ldots, w_{(k)}) = 0 \\ &N(w, v) + \Delta_1(z, w_{(1)}, w_{(2)}, \ldots, w_{(k)}) = 0 \end{aligned} \tag{3.8}$$

where $N(w, v)$ is the linear (with respect to v) part of $\Delta_0(z, u_{(1)}, u_{(2)}, \ldots, u_{(k)})$. Next the symmetry method is applied to the coupled system (3.8). Thus, in this approach, an exact symmetry of the system approximating the original perturbed equation in the first order of precision is referred to as a first-order approximate symmetry of the original equation.

3.2. A new approach

The main points of the new approach are:

(i) The one-parameter (a) Lie group of transformations

$$z^* = f(z; a) \tag{3.9a}$$

$$X = \zeta(z)\frac{\partial}{\partial z} \tag{3.9b}$$

is applied to the *unperturbed equation* $\Delta_0 = 0$ written in the variables u^*, x^* and t^* as

$$\Delta_0(z^*, u^*_{(1)}, \ldots, u^*_{(k)}) = 0 \tag{3.10}$$

which, as the result, is transformed to

$$\tilde{\Delta}_0(z, u_{(1)}, \ldots, u_{(k)}; a) = 0 \tag{3.11}$$

or infinitesimally

$$\Delta_0(z^*, u^*_{(1)}, \ldots, u^*_{(k)}) = \Delta_0(z, u_{(1)}, \ldots, u_{(k)}) + aX^{(k)}\Delta_0(z, u_{(1)}, \ldots, u_{(k)})|_{\Delta_0=0} + \mathrm{O}(a^2) \quad (a \ll 1) \tag{3.12}$$

where $X^{(k)}$ is the kth extended infinitesimal generator of (3.9b).

(ii) The invariance requirement is replaced by the requirement that the unperturbed equation (3.10) transform infinitesimally (for small values of the group parameter a) into the perturbed equation (3.1) with $\epsilon = a$. This requirement may be expressed as

$$X^{(k)}\Delta_0(z, u_{(1)}, \ldots, u_{(k)})|_{\Delta_0=0} = \Delta_1(z, u_{(1)}, \ldots, u_{(k)}). \tag{3.13}$$

It yields determining equations for the group generators $\zeta = (\xi, \eta)$.

(iii) Having the group generators defined, the finite transformations (3.9a) are determined as a solution of the Cauchy problem

$$\frac{\mathrm{d}f(z; a)}{\mathrm{d}a} = \zeta(f) \qquad f(z; 0) = z. \tag{3.14}$$

These transformations are used in (3.10) to define a new equation (3.11), which, in view of (i) and (ii), possesses the following two properties:

(a) When $a \ll 1$, equation (3.11) coincides with the initial perturbed equation (3.1) up to first order in $a = \epsilon$:

$$\tilde{\Delta}_0(z, u_{(1)}, \ldots, u_{(k)}; a) = \Delta_0(z, u_{(1)}, \ldots, u_{(k)}) + a\Delta_1(z, u_{(1)}, \ldots, u_{(k)}) + \mathrm{O}(a^2). \tag{3.15}$$

(b) There exists the exact transformation $z = f(z^*, -a)$ (inverse to (3.9a)) that converts equation (3.11) into the unperturbed equation (3.10). Therefore any exact solution of the unperturbed equation yields the exact solution of equation (3.11).

If an exact solution $u^* = \Phi(x^*)$ of the unperturbed equation (3.10) is known, the approximate solution $u(x)$ of the perturbed equation (3.1) can be obtained by introducing the infinitesimal transformations into the solution $\Phi(x^*)$, as follows:

$$u(x) + \epsilon\eta(x, u(x)) = \Phi(x + \epsilon\xi(x, u(x))) \tag{3.16}$$

with subsequent expanding of the result up to first order in ϵ. Equally, this approximate solution can be obtained from the corresponding exact solution of the new equation (3.11) by expanding it up to first order in a and replacing a by ϵ afterwards.

Thus, the new approach allows one:

(i) To extend any solution of the unperturbed equation to the approximate solution of the perturbed equation. This, in general, provides more possibilities for constructing approximate solutions than an application of the approximate group method—see discussion in the next section.
(ii) To construct equations that on the one hand are integrable (if the unperturbed equation is integrable), and on the other hand have solutions with some prescribed (at least, in some parameter interval) features.

In the next section we use the approach to obtain approximate solutions of the perturbed nonlinear wave equations, that have equation (1.1) as the unperturbed part, and to construct some nonlinear wave equations that can be reduced to (1.1) by an exact transformation.

4. Perturbed nonlinear wave equations

4.1. An example of application of the approach

We will start from the perturbed nonlinear wave equation of the form

$$u_{tt} + \epsilon u_t = (uu_x)_x \tag{4.1}$$

which arises from one-dimensional gas dynamics [13], longitudinal wave propagation on a moving threadline [14] and one-dimensional wave propagation in nonlinear, rate-dependent materials [24]. The approximate classical symmetries of equation (4.1) and the corresponding approximate solutions were discussed by Baikov *et al* [15]. Solutions of equation (4.1) obtained by the extension of the approximate symmetry group method to conditional symmetries were considered in [25].

Following the approach described in section 3.2, we consider the one-parameter (a) Lie group of point transformations

$$x^* = f(x, t, u; a) \qquad t^* = g(x, t, u; a) \qquad u^* = h(x, t, u; a) \tag{4.2a}$$

$$X = \xi(x, t, u)\frac{\partial}{\partial x} + \tau(x, t, u)\frac{\partial}{\partial t} + \eta(x, t, u)\frac{\partial}{\partial u} \tag{4.2b}$$

which convert the unperturbed equation

$$\Delta_0(x^*, t^*, u^*) = u^*_{t^*t^*} - (u^*u^*_{x^*})_{x^*} = 0 \tag{4.3}$$

into another equation $\tilde{\Delta}_0(x, t, u; a) = 0$, such that

$$\tilde{\Delta}_0(z, t, u; a) = \{u_{tt} - (uu_x)_x\} + a\{ru_t\} + O(a^2) \qquad (a \ll 1) \tag{4.4}$$

where r is a trace coefficient. The generators of such a group are determined from the requirement (3.13) which results in the determining equations for ξ, τ and η having the following solutions:

$$\begin{aligned}\xi &= c_4 x + c_2 \\ \tau &= c_3 t + c_1 - r\frac{t^2}{10} \\ \eta &= 2u(c_4 - c_3) + r\tfrac{2}{5}ut\end{aligned} \tag{4.5}$$

where c_1, c_2, c_3 and c_4 are arbitrary constants and the trace coefficient r marks terms additional to those of the symmetry group of equation (4.3).

Next, we will determine the finite transformations (4.2*a*) generated by (4.2*b*) with ξ, τ and η including only the r-terms of (4.5). Solving the corresponding Cauchy problem

$$\frac{\mathrm{d}g(t,u;a)}{\mathrm{d}a} = -\frac{g^2}{10} \qquad \frac{\mathrm{d}h(t,u;a)}{\mathrm{d}a} = \frac{2}{5}gh \qquad g(t,u;0) = t \quad h(t,u;0) = u \tag{4.6}$$

we obtain the transformations in the form

$$x^* = x \qquad t^* = t\left(1 + \frac{at}{10}\right)^{-1} \qquad u^* = u\left(1 + \frac{at}{10}\right)^4. \tag{4.7}$$

Substituting (4.7) into (4.3) yields

$$\tilde{\Delta}_0(x,t,u;a) = u_{tt} - (uu_x)_x + \frac{a}{1+\frac{at}{10}}u_t + \frac{1}{5}\frac{a^2}{(1+\frac{at}{10})^2}u = 0. \tag{4.8}$$

It is seen that the transformed equation (4.8) has the property defined by (4.4): for $a = \epsilon \ll 1$ it coincides with the original equation (4.1) up to first order in a. At the same time, the exact solutions of equation (4.8) can be obtained from exact solutions of the unperturbed equation (4.3) by the transformation inverse to (4.7). Other forms of this equation possessing the same properties can be easily obtained from (4.8) by applying additional transformations simplifying (4.8) as, for example

$$U_{\theta\theta} + aU_\theta + \tfrac{6}{25}a^2 U = (UU_x)_x \tag{4.9a}$$

where

$$U(x,\theta) = 4\left(1 + \frac{at}{10}\right)^2 u(x,t) \qquad \theta = \frac{5}{a}\ln\left(1 + \frac{at}{10}\right) \tag{4.9b}$$

or

$$v_{\pi\pi} + \frac{a}{1+\frac{3a\pi}{5}}v_\pi = (vv_x)x \tag{4.10a}$$

where

$$v(x,\pi) = 16\left(1 + \frac{at}{10}\right)^5 u(x,t) \qquad \pi = \frac{5}{3a}\left[\left(1 + \frac{at}{10}\right)^{-3/2} - 1\right]. \tag{4.10b}$$

For small $a = \epsilon$, these equations differ from (4.1) by the terms of the order of ϵ^2: equation (4.9*a*) by the source term and equation (4.10*a*) by the additional time-dependent term in the coefficient of the perturbation.

4.2. Approximate solutions of equation (4.1)

First, we will consider approximate solutions of (4.1) originating from the classical point symmetries of the unperturbed equation (1.1) which will enable us to compare tools for constructing approximate solutions provided by our approach and by the approximate classical symmetry group method of Baikov *et al* [15]. (It is worth noting here that the applications of Baikov *et al*'s approach are not restricted in finding approximate solutions—they also include calculating approximate conservation laws and approximate symmetry groups of PDEs.) The classical point symmetries of equation (1.1) are represented by (refer to (4.5) for $r = 0$):

$$X_1^{(0)} = \frac{\partial}{\partial t} \qquad X_2^{(0)} = \frac{\partial}{\partial x} \qquad X_3^{(0)} = t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u} \qquad X_4^{(0)} = x\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u}. \tag{4.11}$$

The approximate solutions obtainable by Baikov *et al*'s method [15] are the invariant solutions based on the approximate symmetries of equation (4.1). Applying the approximate symmetry group method to (4.1) yields two approximate symmetries, one of which coincides with the exact symmetry $X_4^{(0)}$ of this and unperturbed equations, and another one is

$$X_3^{(A)} = \left(t + \epsilon\frac{t^2}{10}\right)\frac{\partial}{\partial t} - 2u\left(1 + \epsilon\frac{t}{5}\right)\frac{\partial}{\partial u} \tag{4.12}$$

which is a *stable* symmetry inherited from $X_3^{(0)}$. The approximate similarity variables constructed from (3.6) with the generator (4.12) are

$$\vartheta = x \qquad w = ut^2\left(1 + \epsilon\frac{t}{5}\right) \tag{4.13}$$

so that the approximate invariant solution is

$$u = t^{-2}\left(1 - \frac{\epsilon}{5}t\right)w(x) \tag{4.14a}$$

where $w(x)$ satisfies the equation

$$(ww')' = 6w. \tag{4.14b}$$

The $\epsilon = 0$ counterpart of (4.14) given by

$$u = t^{-2}w(x) \tag{4.15}$$

with $w(x)$ satisfying the same equation (4.14*b*), represents the invariant solution of the unperturbed equation (1.1) corresponding to the unperturbed part $X_3^{(0)}$ of the symmetry $X_3^{(A)}$. Note that other possible invariant solutions of the unperturbed equation are of no use here and thus do not lead to approximate solutions of the perturbed equation.

Let us show first that the same approximate solution (4.14) of equation (4.1) is obtained by applying our approach with the invariant solution (4.15) of the unperturbed equation (1.1) used as a source. To apply the transformations found in section 4.1, we have first to rewrite the source solution (4.15) in variables with stars as

$$u^* = (t^*)^{-2}w(x^*). \tag{4.16}$$

Then we may use in (4.16) the infinitesimal transformations corresponding to the generators (4.5) with ϵ replacing a, as follows:

$$u^* \approx u\left(1 + \epsilon\tfrac{2}{5}t\right) \qquad t^* \approx t - \epsilon\frac{t^2}{10} \qquad x^* = x \tag{4.17a}$$

(we have taken only the r-terms in the generators), which gives

$$u\left(1 + \epsilon\frac{2}{5}t\right) = \left(t - \epsilon\frac{t^2}{10}\right)^{-2} w(x). \tag{4.17b}$$

Solving this for u and expanding the result up to first order in ϵ yields the solution (4.14). Equally the finite transformations (4.7) might be used in (4.16) to obtain u in the form

$$u = t^{-2}\left(1 + \frac{at}{10}\right)^{-2} w(x) \tag{4.18}$$

(it represents the exact solution of equation (4.8)), which turns into (4.14) for small $a = \epsilon$.

While the approximate symmetry group approach produces only the approximate invariant solution (4.14), which is based on the sole approximate symmetry of the perturbed equation (4.1) inherited from the symmetries of the unperturbed equation, our approach allows one to use other symmetries of the unperturbed equation for producing the approximate solutions of (4.1) from the corresponding invariant solutions of (1.1). Let us take, as an example, the symmetry

$$X = c_1 X_1^{(0)} + c_2 X_2^{(0)} = c_1 \frac{\partial}{\partial t} + c_2 \frac{\partial}{\partial x} \tag{4.19}$$

which leads to the invariant solution of (1.1) of the form

$$u = w(\vartheta) \qquad \vartheta = x - Ct \qquad (C = c_2/c_1) \tag{4.20a}$$

$$C^2 w'' = (ww')'. \tag{4.20b}$$

Applying either the infinitesimal (4.17a) or the corresponding finite (4.7) transformations to the solution (4.20a) (written with u^* and t^* replacing u and t) and expanding the result up to first order in $a = \epsilon \ll 1$ produces the approximate solution of equation (4.1) in the form

$$u = (1 - \tfrac{2}{5}\epsilon t) w(\vartheta) \qquad \vartheta = x - Ct + \frac{\epsilon C t^2}{10} \tag{4.21}$$

where $w(\vartheta)$ satisfies the same equation (4.20b).

In general, one can construct approximate solutions of equation (4.1) using any solution of equation (1.1)—for example, a conditional invariant solution. The conditional symmetries of equation (1.1) were considered in [19]. We will take, as an example, the conditional symmetry with the generator

$$V_{2,2} = \frac{\partial}{\partial t} + C_1 t \frac{\partial}{\partial x} + 2C_1^2 t \frac{\partial}{\partial u} \tag{4.22}$$

where C_1 is a constant. The symmetry (4.22) leads to the invariant conditional solution of equation (1.1) having the following form:

$$u = C_1^2 t^2 + w(z) \qquad \vartheta = x - \tfrac{1}{2} C_1 t^2 \tag{4.23a}$$

$$(ww')' = 2C_1^2 - C_1 w' \tag{4.23b}$$

which was discussed in [19,26]. Our approach allows one to construct the approximate solution of equation (4.1) by applying the infinitesimal transformations with the generators (4.5) to the solution (4.23) (rewritten in variables with stars). If we take the generators including, in addition to the r-terms, the c_1- and c_3-terms, the transformations will be

$$u^* \approx u\left[1 + \epsilon\left(-2c_3 + \frac{2t}{5}\right)\right] \qquad t^* \approx t + \epsilon\left(c_1 + c_3 t - \frac{t^2}{10}\right) \qquad x^* = x \tag{4.24}$$

which leads to the approximate solution of the form

$$\begin{aligned} u &= C_1^2 t^2 + w(\vartheta) + \epsilon[2c_1 C_1^2 t + 4c_3 C_1^2 t^2 - \tfrac{3}{5} C_1^2 t^3 + (2c_3 - \tfrac{2}{5} t) w(\vartheta)] \\ \vartheta &= x - \tfrac{1}{2} C_1 t^2 - \epsilon(c_1 C_1 t + c_3 C_1 t^2 - \tfrac{1}{10} C_1 t^3) \end{aligned} \tag{4.25}$$

where $w(\vartheta)$ satisfies equation (4.23b). It is readily verified that the solution (4.25) satisfies equation (4.1) in the first order of precision.

Let now consider some of the approximate solutions of the perturbed nonlinear wave equation (4.1) obtained by applying our approach to the new exact solutions of equation (1.1) that were found in section 2 via potential symmetries. Introducing the transformation (4.17a) into (2.18) rewritten in variables with stars and expanding the result up to first order in a and subsequently replacing a by ϵ yields

$$ux^2 - u^2t^2 \approx b_2\alpha\beta\left(\alpha^2 + \frac{b_1}{2} + \alpha\beta\right)^{-3} + \epsilon t\left\{3u^2t^2 - 2ux^2 + \frac{b_2}{5}\left[\frac{2(\alpha^2 - 6\alpha\beta + \beta^2)}{(\alpha+\beta)^6}\left(\frac{9ux^2 - 13u^2t^2}{R} - 1\right)\right]\right\} \qquad (I = 1) \tag{4.26}$$

and

$$ux^2 - u^2t^2 \approx b_2\alpha\beta\left(\alpha^2 + \frac{b_1}{2} + \alpha\beta\right)^{3} + \epsilon t\left\{3u^2t^2 - 2ux^2 + \frac{b_2}{5}\left[\frac{(\alpha+\beta)^6(\alpha^2 + 6\alpha\beta + \beta^2)}{32}\left(\frac{9ux^2 - 13u^2t^2}{R} - 1\right)\right]\right\} \qquad (I = -1) \tag{4.27}$$

where α, β and R are defined by (2.18b). Each of equations (4.26) and (4.27) defines a family of approximate solutions of equation (4.1) depending on two arbitrary constants b_1 and b_2.

Next, we will obtain the approximate solutions of equation (4.1) by introducing the transformation (4.24) into (2.19) and (2.20) which yields

$$x^2 \approx ut^2 + K\left(\frac{t}{u}\right)^{4/5} + \frac{\epsilon}{5}\left[2K\left(t - 6c_3 - 2\frac{c_1}{t}\right)\left(\frac{t}{u}\right)^{4/5} - ut(10c_1 + t^2)\right] \tag{4.28}$$

and

$$t^2 \approx \frac{x^2}{u} - C\left(\frac{x}{u^2}\right)^{8/7} + \frac{\epsilon}{5}\left[\frac{32}{7}C(5c_3 - t)\left(\frac{x}{u^2}\right)^{8/7} + 10t(c_1 + c_3t) - t^3 + \frac{2x^2}{u}(t - 5c_3)\right] \tag{4.29}$$

where K, C, c_1 and c_2 are arbitrary constants Each of equations (4.28) and (4.29) defines a family of solutions depending on three arbitrary constants.

Note that in the case when the source solutions of the unperturbed equation are in an implicit (or complicated) form, so that some auxiliary calculations are needed to get an idea of how the solutions of the unperturbed equation behave (e.g., to solve a transcendental equation in the case of the solutions (2.18)–(2.20)), the natural way to study the corresponding solutions of the perturbed equation is simply to implement the same calculations with the transformed variables without writing out the perturbed solution.

We will also make remarks about the possible usefulness of the approximate solutions of equation (4.1) obtained in this section. First of all, the solutions (4.26)–(4.29) enable us to study, in a specific physical context, the influence of the effects related to the perturbation term upon the phenomena described by the unperturbed solutions (2.18)–(2.20). For example, in the context of the dynamics and wave propagation of nonlinear dissipative Maxwellian materials [24], the perturbation term in (4.1) may be regarded as the result of incorporating the effects of the relaxation of stress. Then, from the analysis of the approximate solutions of (4.1) stemming from the solutions of (1.1) that describe the propagating discontinuities, it is seen that the effects of the stress relaxation result in a delay of the discontinuity development, lowering the value of u at the discontinuity (at which u_x is infinite) and decreasing the speed of the discontinuity propagation. It should also be remarked upon a general usefulness of the transformations from the unperturbed to perturbed equation that are defined as a result of application of the method. Determining these transformations can be considered as finding some generalized approximate solution to a given perturbed equation since they provide a possibility to built an approximate solution from *any* solution of the unperturbed equation and their forms do not depend on the form of the source solution. As a matter of fact, even a numerical solution of the unperturbed equation can be used as a source to obtain the corresponding approximate solution of the perturbed equation by the transformation of variables.

4.3. Other perturbed equations

The approach developed can be applied to other perturbed nonlinear wave equations, that are more complicated than (4.1), as, for example, the following:

$$u_{tt} + \epsilon[k_0 u_t + k_1 u u_x + k_2 u_{xt} + k_3 (u u_x)_t] = (u u_x)_x \tag{4.30}$$

where k_0, k_1, k_2 and k_3 are constants. One may find approximate solutions of (4.30) using the infinitesimal transformations with the generators

$$\begin{aligned} \xi &= c_4 x + c_2 - k_1 \frac{x^2}{14} - k_2 \frac{t}{2} \\ \tau &= c_3 t + c_1 + k_3 \frac{x}{2} - k_0 \frac{t^2}{10} \\ \eta &= 2u(c_4 - c_3) + k_0 \tfrac{2}{5} u t - k_1 \tfrac{2}{7} u x \end{aligned} \tag{4.31}$$

and one may use the corresponding finite transformations to construct the equations that coincide with (4.30) for small $a = \epsilon$ and are reducible to (1.1) by exact transformations. Considering, as an example, the perturbed equation of the form (4.30) with $k_0 = k_1 = 1$ and $k_2 = k_3 = 0$, we arrive at a new equation, which (after an additional transformation similar to (4.10)) differs from the perturbed equation by the source term proportional to a^2, namely

$$U_{\theta\theta} + a(U_\theta + U U_\chi) + a^2 \left(\tfrac{6}{25} U - \tfrac{6}{49} U^2\right) = (U U_\chi)_\chi. \tag{4.32}$$

Solutions of this equation are related to solutions of equation (4.3) by

$$u^* = U(\chi, \theta) \mathrm{e}^{2a(\theta/5 - \chi/7)} \qquad x^* = \frac{14}{a}(1 - \mathrm{e}^{-a\chi/7}) \qquad t^* = \frac{10}{a}(1 - \mathrm{e}^{-a\theta/5}). \tag{4.33}$$

5. Discussion

In this paper we have investigated the potential symmetries of the nonlinear wave equation (1.1) and corresponding invariant solutions. Despite the complicated structure of the similarity variables we succeeded in determining closed-form solutions of the related ODEs and finding new classes of exact solutions of equation (4.1) this way.

Further we have presented a new approach to the use of the Lie group technique for differential equations depending on a small parameter. As a result, we have found new approximate solutions of the perturbed nonlinear wave equations, in particular, those stemming from the exact solutions of the unperturbed equation defined via potential symmetries. As another result of applying the new method we have constructed some nonlinear wave equations that can be reduced to the unperturbed equation by exact transformations and that approximately, within some range of the equation parameter, coincide with the initial (perturbed) equation. Below we will make several comments on the method developed.

First, we will comment on the new method versus the standard perturbation technique. As distinct from the perturbation methods of solving differential equations (see, e.g., [27]), our method is aimed at finding transformations between different equations. Correspondingly, while the perturbation methods involve the straightforward expansion of the dependent variable inserted into the perturbed equation (sometimes it is accompanied by a transformation of the independent variables as an artificial device), in our method, the transformations from the perturbed equation to the unperturbed equation are sought. Which variables are transformed and in what way is determined by the requirement that the transformations form a Lie group. These transformations naturally define the approximate solution of the perturbed equation. Another feature of our method, that should be of interest for the theory of perturbation methods, is that it produces a new equation, which on the one hand can be transformed to the zero-order equation by an exact transformation and on the other hand has naturally defined first-order approximation.

We will also remark upon the relations between our method and a well-known method of Lie series and transforms of perturbation theory (see [28]). Even though these two methods might seem to be closely related, they, in fact, apply the Lie group ideas for different purposes. In perturbation theory, a near identity transformation is introduced and the Lie transforms (series) are generated to transform a weakly nonlinear (perturbed) system into another weakly nonlinear system that contains long-period terms only (it provides an elegant and algorithmic way to implement calculations in the framework of the method of averaging). In our method, the Lie group technique is used to transform a perturbed equation or system straight into the unperturbed (not obligatory linear) equation or system.

The next comment concerns application of our method to the perturbed equations in which the perturbations contain derivatives of a higher order than that of the unperturbed equation—we will mention, as examples, the well known perturbed Burgers and Korteweg–de Vries (KdV) equations considered in the context of the asymptotic integrability of physical systems in a series of papers (see, e.g., [29–32]). To treat such equations, the approach developed in this paper for point transformations is generalized to include Lie–Bäcklund transformations. Applying our approach with Lie–Bäcklund transformations to the perturbed Burgers and KdV equations provides a unifying group-theoretical framework for different results, that were obtained in [29–32] by using the idea of near identity transformation [33], and can also yield some new results (they will be a subject of a separate publication). In this context, both the infinitesimal transformations and new integrable equations produced by the method may be of interest.

In some cases, application of our method may provide an opportunity to discover integrability of a given equation. As an example, we will consider the Burgers equation which, being written in the integrated form with a small parameter ϵ introduced by rescaling the dependent variable, is

$$\Delta = u_t - u_{xx} + \epsilon u_x^2 = 0. \tag{5.1}$$

To apply our method (section 3.2), we are looking for the one-parameter (a) Lie group of point transformations, which would convert the unperturbed equation written in variables with stars into another equation coinciding for $a = \epsilon \ll 1$ with equation (5.1). Retaining in the group generators only the terms additional to the symmetry group terms we have

$$\xi = 0 \qquad \tau = 0 \qquad \eta = -\frac{u^2}{2} \tag{5.2}$$

from which the finite transformations are obtained in the form

$$x^* = x \qquad t^* = t \qquad u^* = \frac{u}{1 + a\frac{u}{2}}. \tag{5.3}$$

This being substituted into the unperturbed equation gives the new equation

$$\tilde{\Delta}_0 = u_t - u_{xx} + \frac{a}{1 + a\frac{u}{2}} u_x^2 = 0 \tag{5.4}$$

which, in accordance with the basic idea of the method, differs for $a = \epsilon \ll 1$ from (5.1) by the terms of the order of ϵ^2. The structure of this equation suggests using the variable $1 + au/2$ instead of u to make simplifications, after which it appears that the transformation

$$1 + a\frac{u}{2} = \mathrm{e}^{a\rho} \tag{5.5}$$

reduces it to the initial equation. Thus, the inverses of (5.5) and (5.3) define the *exact* transformation between the unperturbed, heat conduction equation and perturbed Burgers' equation which represents the well known Cole–Hopf transformation [34].

Of course this method cannot be systematically used to determine whether an equation is linearizable. Lie group theory provides a more systematic approach to this question that has been developed by Kumei and Bluman in [35] (see also [1]). Upon replacing Burger's equation by its associated system of the first-order PDEs, the Cole–Hopf transformation is derived as an application of theorems 6.4.1-1 and 6.4.1-2 of [1]. Lie group theory provides also a way to discover mapping of a given linear PDE to a specific target PDE by means of comparing the symmetry groups of these two equations (see, e.g., [1]), which being applied to the heat conduction and Burgers' equations leads to the same transformation. We have presented the above example to show that the method designed for other purposes—namely, determining approximate solutions of differential equations with a small parameter and constructing the related integrable equations—may in some cases produce an integrable equation which coincides with the original perturbed equation.

In conclusion, we note that, besides generalizations of the method to contact and Lie–Bäcklund transformations, its modifications in the spirit of the nonclassical method are also possible. No difficulties arise in applying the same approach to ordinary differential equations.

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